Two-stage Interpolation Systems[1]

Maria Paola Bonacina

Dipartimento di Informatica
Università degli Studi di Verona
Verona, Italy

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[1] Joint work with Moa Johansson

Interpolation system

- Given refutation of $A \cup B$ extracts interpolant of (A, B)
- Associates partial interpolant to every clause
- Defined inductively based on those of parents
- Partial interpolant of $\Box$ is interpolant of (A, B)
- Suitable for generic inference systems

Other approaches: interpolation built into decision procedures, interpolation as constraint solving ...

Completeness

- Inference system Γ: set of inference rules to build refutation
- Interpolation system complete for Γ if it extracts an interpolant from any Γ-refutation
- More than one

Requirements

(Reverse) interpolant I of (A, B) such that $A, B \vdash \bot$:

1. $A \vdash I$
2. $I, B \vdash \bot$
3. All uninterpreted symbols in I are shared

State of the art approach

- Track symbols in refutation
- To determine whether a literal should be added to interpolant by checking simultaneously
  - Whether it comes from A or B side of proofs (req's 1 and 2)
  - That all its uninterpreted symbols are shared (req. 3)

Terminology

Uninterpreted symbol:

- A-local or A-colored: in A but not in B
- B-local or B-colored: in B but not in A
- Global or Transparent: in both

Term/atom/literal/clause:

- Transparent: all symbols transparent
- A-colored: all symbols in A and at least one A-colored
- B-colored: all symbols in B and at least one B-colored
- Otherwise: AB-mixed

Partial interpolant

- C occurs in a refutation of $A \wedge B$
- $A \wedge B \vdash C$
- $A \wedge B \wedge \neg C \vdash \bot$
- Interpolant of $A \wedge \neg C$ and $B \wedge \neg C$
- $(A \wedge \neg C, B \wedge \neg C)$ may share more symbols than (A, B)
- Use projections

Projections and Partial Interpolants

- $C|_A$: A-colored and transparent literals of C
- $C|_B$: B-colored and transparent literals of C
- Alternative: transparent in $C|_B$ only
- $\bot$ if empty
- Partial Interpolant of C:
  interpolant of $A \wedge \neg(C|_A)$ and $B \wedge \neg(C|_B)$

Interpolation for propositional/ground resolution

- All literals are input literals
- Either A-colored or B-colored or transparent, no AB-mixed
- Define partial interpolant by case analysis on resolved literal
- [Jan Krajíček 1997, Pavel Pudlàk 1997, Ken McMillan 2003]
- Equality?

Equality

- What if AB-mixed equation $t_a \simeq t_b$ is derived?
- Rewriting:
  t_a (t_b): A-colored (B-colored) ground term in normal form
  $t_a \succ t_b$: replace t_a with t_b everywhere
  t_b should become transparent
- A-colored/B-colored/transparent should change dynamically!

Separating ordering

Ordering $\succ$ on terms and literals:
separating if $t \succ s$ whenever s is transparent and t is not

[Harald Ganzinger et al. 2006, Ken McMillan 2008, Laura Kovàcs and Andrei Voronkov 2009]

Rewriting: t_a and t_b rewritten to t

Lemma: Separating ordering implies no AB-mixed literals in ground superposition proofs

Interpolation system GΓI: ground superposition

C clause in ground Γ-refutation of $A \cup B$:

- $c: C \vee l[r] \vee D$ generated from $p_1: s \simeq r \vee C$ and $p_2: l[s] \vee D$
  - $s \simeq r$ A-colored: $PI(c) = PI(p_1) \vee PI(p_2)$
  - $s \simeq r$ B-colored: $PI(c) = PI(p_1) \wedge PI(p_2)$
  - $s \simeq r$ transparent: $PI(c) = (s \simeq r \vee PI(p_1)) \wedge (s \not\simeq r \vee PI(p_2))$
- Resolution, Superposition into equational literal and Simplification ... treated similarly

Theorem: If the ordering is separating, GΓI is a complete interpolation system for ground Γ-refutations

Interpolation of non-ground proofs?

Inferences apply substitutions creating AB-mixed literals

$g(y, b) \simeq y$ and $f(g(a, x), x) \simeq f(x, a)$, with $\sigma = \{y \leftarrow a, x \leftarrow b\}$, generates $f(a, b) \simeq f(b, a)$, where both sides are AB-mixed literals

This inference is compatible with a separating ordering, which therefore no longer suffices

Restrict to proofs with no AB-mixed literals?

Not enough:

$\neg P(x, b) \vee C$ and $P(a, y) \vee D$ with $x \notin Var(C)$, $y \notin Var(D)$,
$\sigma = \{x \leftarrow a, y \leftarrow b\}$ yield $(C \vee D)\sigma$

$\neg P(a, b)$ and $P(a, b)$ AB-mixed

Case analysis on resolved literal does not suffice

Restrict to proofs with no *AB*-mixed clauses?

So called local inferences: only one color involved

Not enough:

$p_1 : L(x, a) \vee R(x)$ and $p_2 : \neg L(c, y) \vee Q(y)$ with
$\sigma = \{x \leftarrow c, y \leftarrow a\}$ yield $R(c) \vee Q(a)$

$(PI(p_1) \vee PI(p_2))\sigma$ may not be transparent

A two-stage approach

Separate entailment and transparency requirements

First stage: compute provisional interpolant $\hat{I}$
such that $A \vdash \hat{I}$ and $B, \hat{I} \vdash \bot$
but may contain colored symbols

[Guoxiang Huang 1995]

Use labels to track where literals come from

- Clause in A: literals get label **A**
- Clause in B: literals get label **B**
- Resolvent $(C \vee D)\sigma$:
  literals inherit labels from literals in parents
- Paramodulant $(C \vee L[r] \vee D)\sigma$: as for resolution and $L[r]\sigma$ inherit label of para-into literal

Example

$L(x_1, c)_{\mathbf{A}} \vee P(x_1)_{\mathbf{A}} \vee Q(x_1, y_1)_{\mathbf{A}}$

$\neg L(c, x_2)_{\mathbf{B}} \vee P(x_2)_{\mathbf{B}} \vee R(x_2, y_2)_{\mathbf{B}}$

$\sigma = \{x_1 \leftarrow c, x_2 \leftarrow c\}$

Resolvent: $P(c)_{\mathbf{A}} \vee Q(c, y_1)_{\mathbf{A}} \vee P(c)_{\mathbf{B}} \vee R(c, y_2)_{\mathbf{B}}$

which becomes $Q(c, y_1)_{\mathbf{A}} \vee P(c)_{\mathbf{B}} \vee R(c, y_2)_{\mathbf{B}}$ after merging

Labeled Projections and Provisional Partial Interpolants

- $C|_{\mathbf{A}}$: literals of C labeled **A**
- $C|_{\mathbf{B}}$: literals of C labeled **B**
- $\bot$ if empty
- Commute with substitutions: resolvent $(C \vee D)\sigma$
  $(C \vee D)\sigma|_{\mathbf{A}} = (C|_{\mathbf{A}} \vee D|_{\mathbf{A}})\sigma$
- Provisional partial interpolant: provisional interpolant of $A \wedge \neg(C|_{\mathbf{A}})$ and $B \wedge \neg(C|_{\mathbf{B}})$
- Provisional partial interpolant of $\Box$ is provisional interpolant of (A, B)

Provisional interpolation system

- $c : (C \vee D)\sigma$ resolvent of $p_1 : L \vee C$ and $p_2 : \neg L' \vee D$:
  - Both literals **A**-labeled: $\widehat{PI}(c) = (\widehat{PI}(p_1) \vee \widehat{PI}(p_2))\sigma$
  - Both literals **B**-labeled: $\widehat{PI}(c) = (\widehat{PI}(p_1) \wedge \widehat{PI}(p_2))\sigma$
  - Positive **A**-labeled and negative **B**-labeled: $\widehat{PI}(c) = [(L \vee \widehat{PI}(p_1)) \wedge \widehat{PI}(p_2)]\sigma$
  - Vice versa: $\widehat{PI}(c) = [\widehat{PI}(p_1) \wedge (\neg L' \vee \widehat{PI}(p_2))]\sigma$
- Superposition: similar with para-from and para-into literals

A complete provisional interpolation system

- Build provisional interpolant mostly by adding instances of **A**-labeled literals resolved or paramodulated with **B**-labeled ones: *communication interface*
- **Theorem:** Provisional interpolation system is complete
- **Lemma:** Provisional interpolants are in negation normal form with $\forall$-quantified variables and transparent predicate symbols

Second stage: lifting

- All functions in provisional interpolant interpreted or transparent
- Replace *A*-colored constant by ∃-quantified variable
- Replace *B*-colored constant by ∀-quantified variable

A complete interpolation system

- **Theorem:** Lifting of provisional interpolant $\hat{I}$ of (A, B) is interpolant:
  - **Lemma:** $B, \hat{I} \vdash \perp$ implies $B, Lift(\hat{I}) \vdash \perp$
    bwoc: assume $B, Lift(\hat{I})$ has model ...
    A-colored constants new, interpreted as $\exists$-vars
  - **Lemma:** $A, \neg\hat{I} \vdash \perp$ implies $A, \neg Lift(\hat{I}) \vdash \perp$
    bwoc: assume $A, \neg Lift(\hat{I})$ has model ...
    B-colored constants new, interpreted as $\exists$-vars (after negation)
  - $Lift(\hat{I})$ is transparent
- **Corollary:** Provisional interpolation system + lifting = complete interpolation system

Problems with two-stage approach

- Does not handle colored function symbols in provisional interpolant
- Lifting terms with colored top symbol removes too much information
- Other work: proof transformation (localization)
  applied to ground proofs: [Jürgen Christ and Jochen Hoenicke 2010; Ken McMillan 2011; Kryštof Hoder, Laura Kovàcs and Andrei Voronkov 2012]

Summary

- Interpolation systems for first-order theorem proving
- Color-based approach and its limitations
- Two-stage approach and its limitations