The interpolation technique in proof complexity

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1. Proof complexity and its goals

Mathematical logic:
given a proof system P, which formulas are *provable* in P?

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Proof complexity:
given a proof system P, which formulas have *short proofs* in P?

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- The textbook axiomatization of propositional calculus.

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- The textbook axiomatization of propositional calculus.
- Axioms such as

$$A \to A \vee B\,,\ A \to (B \to A)\,,\ldots$$

from which we are supposed to derive a formula by means of the *modus ponens* rule

$$\frac{A\,,\ A \to B}{B}\,.$$

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Theorem (Cook-Reckov)

There exists a polynomially bounded propositional proof system iff $NP = coNP$.

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- There are *many more* systems for which this is not known.

Main open problem: Is the Frege system polynomially bounded?

- Design techniques for proving lower bounds on sizes of proofs.
- Feasible interpolation is one such technique.

2. Some facts about Boolean functions

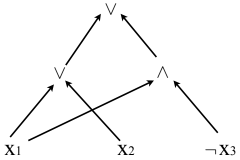
Boolean function - $f : \{0,1\}^n \to \{0,1\}$.

Boolean circuit - directed acyclic graph

- Inputs: in-degree zero

$$x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n.$$

- Operations: $\wedge, \vee$ with in-degree two.



Size - number of nodes/gates.

Circuit size of *f* - the size of a smallest circuit computing f.

Monotone Boolean function

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Examples:

- conjunction, disjunction, the majority function.
- The k-clique function CL_n^k:
  n^2 inputs $\{x_{i,j}; i, j \in [n]\}$ represent edges of a graph on n vertices.

$$\mathrm{CL}_n^k = 1$$

iff the graph has a clique of size k.

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Monotone boolean circuit is a boolean circuit which doesn't contain negations.

- Every monotone function can be computed by a monotone circuit.

Theorem (R,AB)

Set $k := \lceil \sqrt{n} \rceil$. Every monotone circuit computing the clique function CL_n^k must have size at least $2^{\Omega(n^{1/4})}$.

3. Craig's interpolation theorem

x, y, z - disjoint sets of variables.

Craig's interpolation theorem

Assume that $A(x, y) \to B(x, z)$ is a tautology.
Then there exists a formula $C(x)$ such that both
$A(x, y) \to C(x)$, $C(x) \to B(x, y)$ are tautologies.

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C is an *interpolant* of A and B.

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Define

$$C(x) := \bigvee_{\sigma \in \{0,1\}^k} A(x, \sigma), \text{ where } k = |y|.$$

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That is $C(x) = 1$ iff there exists $\sigma \in \{0, 1\}^k$ with $A(x, \sigma) = 1$.

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Or dually,

$$C'(x) := \bigwedge_{\sigma \in \{0,1\}^s} B(x, \sigma)\,, \text{ where } s = |z|.$$

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A symmetric version: if $A'(x, y) \vee B'(x, z)$ is a tautology then there exists $C(x)$ such that

$$C(x) \to A'(x, y)\,,\; \neg C(x) \to B'(x, z)$$

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Monotone interpolation theorem *If $A(x, y)$ is monotone in x then there exists an interpolant $C(x)$ which is monotone.*

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$y = y_{j,i}, j \in [k], i \in [n]$.
$\text{Clique}_n^k(x, y)$ is the conjunction of the following:

1. $\bigvee_{i \in [n]} y_{j,i}$, for every $j \in [k]$,
2. $\neg y_{j_1,i} \vee \neg y_{j_2,i}$, for every $j_1 \neq j_2 \in [k]$, $i \in [n]$,
3. $\neg y_{j_1,i_1} \vee \neg y_{j_2,i_2} \vee x_{i_1,i_2}$, for every $j_1, j_2 \in [k], i_1, i_2 \in [n]$.

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Then:

$$\text{Clique}_n^{k+1}(x, y) \rightarrow \neg\text{Color}_n^k(x, z)$$

is a tautology.

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An interpolant $C(x)$: tautologies

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1. if x has $k+1$-clique then $C(x) = 1$,
2. if x is k-colorable then $C(x) = 0$.

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Let $k := \lceil\sqrt{n}\rceil$. Every monotone interpolant of $\text{Clique}_n^{k+1}(x,y)$ and $\neg\text{Color}_n^k(x,z)$ requires monotone circuit of size at least $2^{\Omega(n^{1/4})}$.

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- In contrast, there exists an interpolant which can be computed a polynomial size *non-monotone* circuit [L].

4. Feasible interpolation

A propositional proof system P has feasible interpolation, if there exists a polynomial q such that: for every implication $A(x, y) \rightarrow B(x, z)$, if it has a proof in P of size s then $A(x, y)$ and $B(x, z)$ have an interpolant of circuit size $\leq q(s)$.

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Theorem

1. *If P has monotone feasible interpolation then the clique vs. coloring tautologies require exponential size proof in P. Hence P is not polynomially bounded.*
2. *If P has feasible interpolation, then it is not polynomially bounded assuming a conjecture in cryptography.*

5. Feasible interpolation for Resolution

Resolution

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A resolution refutation of a set of clauses $\mathcal{C}$ is a sequence of clauses $D_1, \ldots, D_s$ such that

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2. D_s is the empty clause $\emptyset$.

If $A = C_1 \wedge \cdots \wedge C_m$ is a CNF formula, a resolution refutation of A is a refutation of $C_1, \ldots, C_m$.

Resolution

Proposition

Let A be CNF formula. The following are equivalent:

1. *A has a resolution refutation.*
2. *A is not satisfiable (= $\neg A$ is a tautology).*

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Resolution has both feasible interpolation and monotone feasible interpolation.

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In words, assume that $A(x, y) \wedge B(x, z)$ has a resolution refutation of size s.

Then there exists a circuit $C(x)$ of size $O(s)$ such that for every assignment σ to the variables x

- if $C(\sigma) = 0$ then $A(\sigma, y)$ is unsatisfiable,
- if $C(\sigma) = 1$ then $B(\sigma, z)$ is unsatisfiable.

Moreover, if A is monotone in x then C can be taken monotone.

Theorem (Krajíček)

Resolution has both feasible interpolation and monotone feasible interpolation.

Corollary

For $k = \lceil \sqrt{n} \rceil$*, every resolution refutation of* $Clique_n^{k+1} \wedge Color_n^k$ *has size at least* $2^{\Omega(n^{1/4})}$*.*

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- We want a circuit s.t. $(C(\sigma) = 0) \rightarrow A(\sigma, y)$ unsat. $(C(\sigma) = 1) \rightarrow B(\sigma, z)$ unsat.
- Instead, construct such an *algorithm*.

Proof outline

- Let R be a resolution refutation of $A(x, y) \wedge B(x, z)$ of size s.
- We want a circuit s.t. $(C(\sigma) = 0) \rightarrow A(\sigma, y)$ unsat. $(C(\sigma) = 1) \rightarrow B(\sigma, z)$ unsat.
- Instead, construct such an *algorithm*.

Claim 1. For every σ, $A(\sigma, y) \wedge B(\sigma, z)$ has a refutation R_σ of size $\leq s$. R_σ can be constructed from R in polynomial time.

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Claim 2. If A' and B' have disjoint variables then every refutation of $A' \wedge B'$ contains a refutation of A' or a refutation of B'.

6. Feasible interpolation for Cutting Planes

Cutting Planes

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- Manipulates linear inequalities with integere coefficients, $a_1x_1 + \ldots a_nx_n \geq b$, with $a_1, \ldots, a_n, b \in \mathbb{Z}$

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The rules are:

$$\frac{L \geq b}{cL \geq cb}\,,\ \text{if } c \geq 0\,,\ \frac{L_1 \geq b_1\,,\ L_2 \geq b_2}{L_1 + L_2 \geq b_1 + b_2}\,,$$

$$\frac{a_1 x_1 + \ldots a_n x_n \geq b}{(a_1/c)x_1 + \ldots (a_n/c)x_n \geq \lceil b/c \rceil}\,,\ \text{provided } c \text{ divides every } a_i\,.$$

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If $A = C_1 \wedge \ldots C_m$ is a CNF formula, a Cutting Planes refutation of A is a refutation of the inequalities

$$\text{“}C_1\text{”}, \ldots, \text{“}C_m\text{”},\ x_1 \geq 0\,,\ 1 - x_1 \geq 0\,,\ x_2 \geq 0\,,\ 1 - x_2 \geq 0\,, \ldots$$

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Proposition

Let A be a CNF formula. The following are equivalent:

1. *A has a Cutting Planes refutation.*
2. *A is unsatisfiable.*

Monotone real circuit

- computes a boolean function $f : \{0,1\}^n \to \{0,1\}$.
- the inputs as well as the output are in $\{0,1\}$, but
- the intermediary gates can compute an *arbitrary* monotone real function (in two variables).

Theorem (Pudlák)

Cutting planes has monotone interpolation via real monotone circuits.

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In words, assume that $A(x, y) \wedge B(x, z)$ has CP refutation of size s and A is monotone in x.
Then there exists a monotone real circuit $C(x)$ of size $O(s)$ such that for every assignment σ to the variables x

- if $C(\sigma) = 0$ then $A(\sigma, y)$ is unsatisfiable,
- if $C(\sigma) = 1$ then $B(\sigma, z)$ is unsatisfiable.

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Corollary

$Clique_n^{k+1} \wedge Color_n^k$ requires exponential size CP refutations.

7. No feasible interpolation for Frege system

The Frege system

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Axioms:

$$A \to (B \to A)\,,$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))\,,$$
$$(\neg A \to \neg B) \to ((\neg A \to B) \to C)\,,$$
$$A \wedge B \to A,\ A \wedge B \to B\,,$$
$$A \to (B \to A \wedge B)\,,$$
$$A \to A \vee B,\ B \to A \vee B\,,$$
$$(A \to C) \to ((B \to C) \to (A \vee B \to C))\,.$$

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- different textbook axiomatization are equivalent in terms of proof length.
- Frege system is a robust and powerful proof system.

Theorem (B, BPR)

1. *The implication* $Clique_n^{k+1} \rightarrow \neg Color_n^k$ *has a polynomial size Frege proof. Hence Frege system doesn't have monotone feasible interpolation.*

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Theorem (B, BPR)

1. *The implication* $Clique_n^{k+1} \rightarrow \neg Color_n^k$ *has a polynomial size Frege proof. Hence Frege system doesn't have monotone feasible interpolation.*
2. *Assuming that "factoring of integers is hard", Frege system does not have feasible interpolation.*

Sketch of 1.
Frege system can effectively prove the Pigeonhole principle:

"if $f : [k+1] \rightarrow [k]$ *is a total function then there exist* $i \neq j \in [k+1]$ *with* $f(i) = f(j)$*".*

Sketch of 2.

- A one-to-one function f is called *one-way*, if f is easy to compute but the inverse f^{-1} is hard.

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- It is not known whether such functions exist, but this assumption is central to theoretical cryptography.
- Security of encryption schemes such as RSA relies on the existence of one-way functions.

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- If f^{-1} is hard, C must have large circuit (at least for some i).
- [BPR] construct an f which is believed to be one-way (assuming that factoring is hard), but $A(x, y) \to \neg B(x, z)$ has short Frege proof.

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Question: Does Frege system have at least some form of feasible interpolation?
Do Frege proofs have a computational content?

8. Non-clasical logics

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- Does not have the excluded-middle principle $A \vee \neg A$, and does not allow reasoning by contradiction, $\neg\neg A \rightarrow A$.
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Modal logic

- Obtained from propositional logic by adding the symbol $\Box$, where $\Box A$ is intended to mean "A is necessarily true".
- $\Box$ can be understood as "provable in a theory T" (such as Peano arithmetic), leading to provability logic of T.

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- Other systems of modal logic are obtained by adding axioms such as $\Box A \to A$ or $\Box A \to \Box\Box A$.

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1. *$A(\Box x, y) \to \Box(B(x, z))$ is K-tautology.*
2. *If the tautology has a K-proof of size s then there exists a monotone interpolant of A and B of size $O(s^2)$.*

Thank you